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Note

Two-transitive groups on a hyperbolic unital ☆

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Abstract

A classification given previously of all projective translation planes of order q^2 that admit a collineation group G admitting a two-transitive orbit of $q+1$ points is applied to show that the only projective translation planes of order q^2 admitting a hyperbolic unital acting two-transitively on a secant are the Desarguesian planes and the unital is a Buekenhout hyperbolic unital.

Keywords: Two-transitive; Subline orbit

1. Introduction

In 1976, Buekenhout [4] proved that any translation plane of order q^2 with spread in $PG(3,q)$ admits a unital. In fact, the embedded unital is parabolic in the general case, in the sense that the line at infinity of the associated translation plane is a tangent line to the unital. The first main unanswered problem was to determine translation planes that can admit a transitive parabolic unital. By this, it is meant that there is a collineation group of the translation plane fixing the parabolic point and acting transitively on the remaining q^3 points of the unital. Recently, John-

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son [5] was able to show that any semifield plane with spread in $PG(3,q)$, does, in fact, admit a transitive parabolic unital.

Apart from parabolic unitals, some translation planes with spreads in $PG(3,q)$ admit 'hyperbolic unitals,' in the sense that the line at infinity contains $q+1$ points of the unital. The necessary condition is that the translation plane contains a regulus in its spread. Let U be a hyperbolic unital embedded in a translation plane π of order q^2 with spread in $PG(3,q)$. If the unital is of Buekenhout type, that is to say, U arises from a quadric in $PG(4,q)$, then in this setting, U has $q+1$ points on the line at infinity that form the infinite points of a regulus net with the translation plane. It is shown in Johnson and Pomareda [6] that any collineation subgroup of $SL(2,q)\circ SL(2,q)$, where the product is a central product, of π that leaves invariant the regulus net can be arranged to leave invariant the hyperbolic unital.

For example, any conical flock plane admits an elation group of order q that leaves a regulus invariant, fixing one component and acting regularly on the remaining components, and hence the elation group E acts also on the hyperbolic unital. Similarly, the planes of Johnson and Prince [7] of order 81 admitting $SL(2,5)$ leaving invariant a regulus are of this type as well; the group leaves invariant a hyperbolic unital of Buekenhout type. In the first situation there is a group E fixing a point and transitive on the remaining q of the $q+1$ points, and in the second case, various of the planes of order 81 admit groups transitive on the $q+1$ points but not two-transitive. Of course, the Desarguesian plane admits $SL(2,q)$ acting two-transitively on $q+1$ points of a regulus, and the question is whether there are other translation planes admitting groups acting two-transitively on a set of $q+1$ points on the line at infinity that preserve a unital. So, the problem considered in this article is as follows.

Problem 1. *Determine the translation planes of order q^2 and spread in $PG(3,q)$ that admit a hyperbolic unital with a secant line at infinity such that the translation plane admits a doubly transitive group on the secant line. Furthermore, once the planes are determined, determine the unital.*

Actually, in a previous paper, the authors considered a generalization of this problem and considered translation planes of order q^2 without reference to the kernel of the plane and without reference to a unital that admit doubly transitive groups on a set of $q+1$ points on the line at infinity.

Problem 2. *Classify the translation planes of order q^2 admitting a collineation group acting two-transitively on a set of $q+1$ points on the line at infinity.*

Of course, included in such a putative classification would be the Ott–Schaeffer planes and the Hering planes as well as the Desarguesian, since these admit $SL(2,q)$ acting transitively on $q+1$ points, even though these points do not always belong to a regulus. But, note again that we are not making any assumptions on the dimension of the associated vector space; we do not require the spread to initially be in $PG(3,q)$.

The main results of Biliotti, Jha, Johnson, Montinaro [1,3] are:

Theorem 1. *(See Biliotti, Jha, Johnson, Montinaro [1,3].) Let π be a translation plane of order q^2 that admits a collineation group G inducing a two-transitive group on a set Γ of $q+1$ infinite points of π. Then π is one of the following types of planes*:

(1) *Desarguesian,*
(2) *Hering,*
(3) *Ott–Schaeffer,*
(4) *Hall of order* 9.

If we are to consider the nature of a translation plane, it might be asked why the set of $q+1$ points should be required to lie on the line at infinity. Could there be translation planes π, whose projective extensions π^+ admit such groups? Also, must the $q+1$ points necessarily be restricted to a line? For example, could the set Γ of $q+1$ points be an arc or a unital design?

Let ρ^+ be a projective Desarguesian plane of order h^2. Then ρ^+ contains a classical unital U_h of absolute points and non-absolute lines of a unitary polarity that admits a collineation group G isomorphic to $PSU(3,h)$. In this setting U_h has h^3+1 points. The plane ρ^+ is isomorphic to $PG(3,h^2)$ and may be coordinatized by a field isomorphic to $GF(h^2)$. Choose any cubic extension $GF(h^6)$ of this field and consider the Desarguesian plane π^+ isomorphic to $PG(3,h^6)$ coordinatized by $GF(h^6)$. In this case, for $q=h^2$, there is a group extension $PSU(3,q)$ of $PSU(3,h)$ and a unital U_q of q^3+1 points containing U_h. So, there is a projective extension of an affine translation plane of order q^2 that admits a set of $q+1$ points and a set of

$$h^4+h^2+1-\left(1+h^3\right)=h^4-h^3+h^2$$

lines that form a unital design Γ admitting a collineation group isomorphic to $PSU(3,h)$ acting two-transitively on Γ, namely the Desarguesian plane of order q^2.

Similarly, if γ^+ is a projective Desarguesian plane of order h^3, let C_h be a conic of h^3+1 points of γ^+. Then there exists a collineation group isomorphic to $PSL(2,h^3)$ acting two-transitively on the points of C_h. Let π^+ be a projective Desarguesian plane of order h^6 and C_q be a conic of h^6+1 points admitting $PSL(2,h^6)$. Letting $h^6=q^2$, then C_h is an arc of $q+1$ points admitting a collineation group of π^+ isomorphic to $PSL(2,q)$.

Consider a projective plane of order q^2 that contains a set Γ of $q+1$ points admitting a collineation group G inducing a two-transitive group on Γ. Is it possible to determine the projective plane and the group G?

Problem 3. *Let π be a translation plane of order q^2 and let π^+ be the projective extension of π. Let Γ be a set of $q+1$ points of π^+. Determine the projective translation planes π^+ that admit doubly transitive collineation groups acting on a set Γ of $q+1$ points of π^+.*

In another related article [2], the authors consider what might be called the affine part of the problem (where Γ is affine) and, combining results, are able to resolve the more general problem.

The main results of Biliotti, Jha, Johnson, Montinaro [1–3] are:

Theorem 2. *(See Biliotti, Jha, Johnson, Montinaro [1–3].) Let π be a translation plane of order q^2 and let π^+ denote the associated projective extension. Let π^+ admit a collineation group G inducing a two-transitive group on a set Γ of $q+1$ points of π^+. Then π is one of the following types of planes*:

(1) *Desarguesian,*
(2) *Hering,*
(3) *Ott–Schaeffer,*
(4) *Hall of order* 9.

Theorem 3. *(See [2].) Let π^+ be a projective Desarguesian plane of order q^2 containing a set Γ of $q+1$ points upon which there is a collineation group of π^+ inducing a two-transitive group on Γ. Then one of the following occurs.*

(1) *Γ is a set of $q+1$ points on a line L. If π is the affine restriction by L then there is an affine group isomorphic to $SL(2,q)$ of π inducing $PSL(2,q)$ on Γ.*
(2) *Γ is a unital design of $q+1$ points and there is a collineation group isomorphic to $PSU(3, \sqrt[3]{q})$ of π^+ induced on Γ.*
(3) *Γ is an arc contained in a conic and there is a collineation group isomorphic to $PSL(2,q)$ of π^+ induced on Γ.*

Returning to the impetus for this article, one could now ask which of these classes of translation planes admit hyperbolic unitals U equipped with a group G leaving U invariant and acting two-transitively on the points of the infinite secant line of U. Since the Desarguesian plane does admit such groups, acting for example on the Buekenhout hyperbolic unital, the question remains whether the Hering or Ott–Schaeffer planes can admit such unitals and group actions. We are able to show that the action of the group $SL(2,q)$ acting on the Hering or Ott–Schaeffer planes prohibits such group actions on unitals. Now, the question of the nature of the unital then becomes of interest. However, we are able to resolve this question as well, proving the unital is classical (Buekenhout). Indeed, one may formulate the problem in the projective extension of the affine translation plane for a stronger theorem. Our main result, therefore, related to unitals is as follows.

Theorem 4. *Let π be a translation plane of order q^2 and let π^+ denote the projective extension. Assume that π^+ contains a unital U and admits a collineation group G that leaves U and a secant line L invariant and acts two-transitively on $L \cap U$. Then π is Desarguesian and the unital is classical.*

2. The Hall plane of order 9

By the analysis of the authors [3], it is shown that the only doubly transitive action on a set of 4 points on a projective Hall plane of order 9, occurs on the line at infinity and the group G is generated by a pair of Baer collineation groups of order 3 that fix two components, say $x=0$, $y=0$ and has two orbits of length 4 on the remaining points on the line at infinity on which G induces a doubly transitive group $PSL(2,3)$. The unique involution in G, isomorphic to $SL(2,3)$, is the kernel involution. Hence, if there is a doubly transitive action on a secant to a hyperbolic unital, it follows that the secant must be one of the two orbits of length 4. However, G must fix two other infinite points, say P and Q. There are 4 tangent lines incident with P to the unital which are necessarily permuted by G, assuming such an doubly transitive action exists on a unital. The normal 2-group of G of order 8 therefore has an element of order 2 that fixes one of the tangents to P and hence must fix the unique unital point on this tangent. Since there is a unique involution σ, the kernel involution, it follows that σ fixes exactly one affine point. However, σ must permute the remaining 3 tangents to P, and therefore fixes one and fixes the unital point on that tangent, a contradiction. Hence, the Hall plane of order 9 does not admit a collineation group fixing a unital and acting doubly transitive on a secant.

3. The Hering plane

If π is a translation plane of order q^2 and Γ is a set of $q+1$ infinite points upon which a collineation group of π induces a two-transitive group, we know by Theorem 1 that the plane is either Desarguesian, Ott–Schaeffer or Hering and we have a group fixing an affine point 0 isomorphic to $SL(2,q)$.

Now assume originally we have a group G that leaves invariant a hyperbolic unital and acts two-transitively on the secant points on the line at infinity. We now do not know a priori that G fixes an affine point. In spite of this we may show that the Hering plane cannot occur.

Theorem 5. *The Hering plane cannot admit a collineation group G leaving invariant a hyperbolic unital and acting transitively on the points of a secant line at infinity.*

Proof. The full collineation group of the Hering plane has three orbits on the line at infinity of lengths $q+1$, $(q^2-q)/2$, $(q^2-q)/2$. There is a collineation group of order $2(q+1)$ that fixes two points on the line at infinity, one in each of the two orbits of lengths $(q^2-q)/2$. If such a collineation group G exists preserving a hyperbolic unital, then the secant line must be the orbit Γ of length $q+1$. This means that none of the remaining points on the line at infinity are in the unital. Choose a non-unital point P on the line at infinity. There are $q+1$ tangents on each such infinite point (the feet of the infinite point P). The group of order $2(q+1)$ that fixes P permutes these $q+1$ lines. Since q is odd and $2(q+1)_2 > (q+1)_2$, there is a collineation g that must be of order 2 that fixes a tangent line (recall that for group of order k, then k_2 denotes the number of elements of order a power of 2). Then, g must fix the unique affine unital point 0 of the tangent line. Hence, g is an involution in the translation complement of the Hering plane, and is therefore in $\Gamma L(4,q)$. Since q is a non-square, the involutions are in $GL(4,q)$. However, it is known that the only involution in the Hering plane in the translation complement is the kernel involution. Hence, there is a unique affine point fixed by $\langle g\rangle$. There are q^3-q-1 remaining unital points permuted semi-regularly by g, a contradiction. Hence, the Hering plane does not admit such a collineation group. □

Thus we have established the following theorem.

Theorem 6. *A translation plane of order q^2 that admits a collineation group that fixes a unital and acts two-transitively on a secant line is either Desarguesian or Ott–Schaeffer.*

4. The Ott–Schaeffer plane

In this section, we show that the Ott–Schaeffer plane does not occur and we resolve the nature of the unital.

Part of the following analysis is from Johnson and Pomareda [6]. Consider $PG(4,q)$, and let

$$Q_4 : Q_4(x_1,x_2,y_1,y_2,z) = x_1y_2 - x_2y_1 - \gamma z^2,$$

for γ a constant, be a non-degenerate quadric in $PG(4,q)$. Note that if $\Sigma_3 \simeq PG(3,q)$ is given by $z=0$, then $Q_4 \cap \Sigma_3$ is a regulus in Σ_3. Consider the mapping

$$e_\alpha : (x_1,x_2,y_1,y_2,z) \mapsto (x_1,x_2,x_1\alpha+y_1,x_2\alpha+y_2,z), \quad \text{for } \alpha \in GF(q).$$

Since

$$x_1(x_2\alpha + y_2) - x_2(x_1\alpha + y_1) - \gamma z^2 = 0,$$

if

$$x_1 y_2 - x_2 y_1 - \gamma z^2 = 0,$$

we see that there is a group

$$\langle e_\alpha; \alpha \in GF(q)\rangle$$

acting on Q_4 and leaving invariant Σ_3. Consider a Desarguesian spread S of the general form

$$x = 0, \qquad y = xm; \quad m \in GF(q^2),$$

written in Σ_3 and realized affinely in $AG(4, q)$, i.e., over the associated 4-dimensional K-vector space V_4. Noting that

$$x = 0, \qquad y = x\alpha; \quad \alpha \in GF(q)$$

is a regulus R, and writing $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $x\alpha = (x_1\alpha, x_2\alpha)$, we see that S admits an affine elation group in the associated Desarguesian affine plane Σ_S corresponding to $\langle e_\alpha\rangle$ that fixes pointwise one component $x = 0$ of R and acts transitively on $R - \{x = 0\}$. Similarly, consider the mapping

$$g_\alpha : (x_1, x_2, y_1, y_2, z) \mapsto (x_1 + y_1\alpha, x_2 + y_2\alpha, y_1, y_2, z)$$

and similarly note that g_α leaves Q_4 and Σ_3 invariant and the associated group induces an affine elation group fixing the component $y = 0$ pointwise of R and acting transitively on $R - \{y = 0\}$. Hence, we see that we may obtain a group isomorphic to $SL(2, q)$ fixing $0 = (0, 0) = (0, 0, 0, 0)$ generated by elations and 0 is not a point of the unital. Similarly, no point of the partial spread of components incident with 0 of the regulus net R are points of the unital. Hence, the 0-components of R are tangent lines so the 0-components not in R therefore are secant lines and there are $q + 1$ points of the unital on each 0-component not in R. The stabilizer of a component L not in R has order $q + 1$ and is transitive on the unital points on L. To see this, assume that g fixes a component L and fixes a unital point T. We note that if (x_1, x_2, y_1, y_2) is a unital point then $\alpha(x_1, x_2, y_1, y_2)$ is a unital point, for $\alpha \in GF(q)$, if and only if $\alpha = \pm 1$. This is to say that the unital points of T lie in $(q + 1)/(2, q + 1)$ different 1-dimensional $GF(q)$-subspaces. Hence, if g fixes a unital point T and q is even then g fixes q points of the 1-dimensional $GF(q)$-space containing T and there are then q other unital points. If g has order dividing $q + 1$ then g must fix another unital point (at least an element in the group generated by g of order a prime power order must), implying that g is an affine homology. However, there are no affine homologies in $SL(2, q)$. If q is odd and g fixes a unital point T then g fixes q points in the 1-dimensional subspace containing T and permutes the remaining $q - 1$ points. If the order of g is 2 then g is the kernel involution, a contradiction since g fixes at least two affine points. Hence, some power of g fixes another unital point and hence g becomes an affine homology, a contradiction as before. Hence, $SL(2, q)$ acts sharply transitively on the affine points of the unital. Consider $GL(2, q)$ acting on the regulus R. $GL(2, q)$ has $q - 1$ orbits each of which is an $SL(2, q)$-orbit. It follows that each of these orbits may be considered the affine points of a unital sharing the secant line. Hence, we have proved the following theorem.

Theorem 7. *Let π be an affine Desarguesian plane of order q^2 and let $SL(2, q)$ be a collineation group of π fixing an affine point* 0*, leaving invariant a unital and acting two-transitively on a secant line on the line at infinity.*

(1) *Then $SL(2,q)$ is sharply transitive on the affine unital points.*
(2) *There is a set of $q-1$ unitals mutually disjoint on the affine points and sharing the secant line at infinity. These unitals are permuted transitively by $GL(2,q)$ and their affine parts form a partition of the affine points of π that do not lie on the partial spread of components incident with 0 of the regulus defined by the secant line.*

Proof. Just note that there are $q^4-((q^2-1)(q+1)+1)=N$ affine points not on the regulus partial spread: $N=q^4-q^3-q^2+q=q^3(q-1)-(q(q-1))=(q-1)(q^3-q)$, and the number of affine points of each unital is q^3-q. □

Corollary 1. *Let π be an affine Desarguesian plane of order q^2 admitting a hyperbolic unital U with secant line on the line at infinity. If there is a collineation group G of π that leaves U invariant and acts two-transitively on the points of the secant line at infinity then the unital U is classical.*

Proof. We know that we may assume that G is isomorphic to $SL(2,q)$. If G fixes an affine point 0 which is not a unital point then G is generated by elations just as in our previous discussion. This would then force G to act sharply transitively on the affine unital points U. However, this would then force U to be a classical unital by uniqueness of the group $SL(2,q)$ that acts on the secant line; the secant line must define a regulus in a Desarguesian plane by virtue of the collineation group. Take two tangent lines L and M to distinct infinite points of the secant line Γ to the unital. These tangents are unique and therefore fixed by G. Hence, $L\cap M$ is fixed by G and cannot be a point of the unital since this point is affine. This completes the proof of the corollary. □

5. Proof of the main theorem

Proof of Theorem 4. Assume the conditions of the statement of Theorem 4. If the secant line L is then not our original line at infinity ℓ_∞, then the intersection point, say (∞), is left invariant by the group G, and this point cannot be a unital point. Furthermore, this means that the line at infinity is a tangent line and since G leaves the unital U invariant, this implies that G fixes the tangent point (0) on ℓ_∞ as well as (∞). However, in any case, the authors consider the more general affine situation in [2] and only the projective Desarguesian plane is possible, under these assumptions.

Hence, as from our previous sections it is known that we need only consider the Ott–Schaeffer case, the group G isomorphic to $SL(2,q)$ and acting on the unital U has a fixed affine point 0 not in the unital. However, the argument on tangents in the Desarguesian case shows that if L and M are the unique tangent lines incident with distinct points on the secant line at infinity then $L\cap M$ is fixed by G. Since G fixes a point 0, it follows that the 0-components on $PG(1,q)$ are tangent lines. Hence, each 0-component outside this regulus net is a secant line and therefore intersects the unital U in $q+1$ points, none of which is 0. Now if the plane is Ott–Schaeffer and there is a collineation group G that preserves a unital and acts two-transitively on the points of a secant line at infinity, then by the structure of the collineation group of the Ott–Schaeffer plane, there are component orbits of lengths $q+1$ and $(q^2-q)/2$, $(q^2-q)/2$. Then clearly the secant line of the unital is the unique orbit of length $q+1$ and in the Ott–Schaeffer case this set defines a regulus. An involution fixing L will fix exactly one of these points—one on each of q lines not in the regulus net—and $q/2$ in each long orbit. Let S_2 be a Sylow 2-subgroup—these $q/2$ points in each long orbit will be permuted by S_2. There are $q-1$ non-trivial involutions each fixing

$q/2$ different points in each long orbit, giving $(q/2)(q-1)$. The normalizer of order $q-1$ will permute these $q/2$ points and can't fix any, a contradiction. Hence, the Ott–Schaeffer case cannot occur. This completes the proof of the main theorem. □

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